Direction: This homework is due on November 11, 2005. In order to receive full credit, answer each problem completely and must show all work.

1. Let $H = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$ be a subgroup of the additive group of integers $\mathbb{Z}$. Find all the distinct left cosets of $H$ in $\mathbb{Z}$. What is the multiplication table of the factor group $\mathbb{Z}/H$?

Answer: Since $H = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$ can be written as $H = 3\mathbb{Z}$, the distinct left cosets of $H$ in $\mathbb{Z}$ are: $0 + 3\mathbb{Z}$, $1 + 3\mathbb{Z}$ and $2 + 3\mathbb{Z}$. The multiplication table of the factor group $\mathbb{Z}/H$ is given by

\[
\begin{array}{c|cccc}
+ & 0 + 3\mathbb{Z} & 1 + 3\mathbb{Z} & 2 + 3\mathbb{Z} \\
0 + 3\mathbb{Z} & 0 + 3\mathbb{Z} & 1 + 3\mathbb{Z} & 2 + 3\mathbb{Z} \\
1 + 3\mathbb{Z} & 1 + 3\mathbb{Z} & 2 + 3\mathbb{Z} & 0 + 3\mathbb{Z} \\
2 + 3\mathbb{Z} & 2 + 3\mathbb{Z} & 0 + 3\mathbb{Z} & 1 + 3\mathbb{Z}
\end{array}
\]

2. Find all the distinct left cosets of the subgroup $H = \{1, 11\}$ in $U(30)$. What is the Cayley table of the factor group $U(30)/H$?

Answer: Since the order of $U(30)$ is $\phi(30) = \phi(2 \cdot 3 \cdot 5) = 30\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 8$, and the order of $H$ is 2, therefore the number of distinct cosets of $H$ in $U(30)$ is 4. The group $U(30)$ consists of the following elements

$$U(30) = \{1, 7, 11, 13, 17, 19, 23, 29\}.$$ 

Therefore the four distinct cosets of $H$ in $U(30)$ are: $1H = \{1, 11\}$, $7H = \{7, 17\}$, $13H = \{13, 23\}$, and $19H = \{19, 29\}$. The multiplication table of the factor group $U(30)/H$ is given by

\[
\begin{array}{c|cccc}
\cdot_{30} & 1H & 7H & 13H & 19H \\
1H & 1H & 7H & 13H & 19H \\
7H & 7H & 19H & 1H & 13H \\
13H & 13H & 1H & 19H & 7H \\
19H & 19H & 13H & 7H & 1H
\end{array}
\]

3. What is the order of the group $U(1617)$? Express $U(1617)$ as an external direct product of cyclic additive groups of the form $\mathbb{Z}_n$. 

\[
\begin{aligned}
\end{aligned}
\]
Answer: Using Fundamental Theorem of Arithmetic, we have $1617 = 3 \cdot 7^2 \cdot 11$. The order of $U(1617)$ is given by

$$|U(1617)| = \phi(1617) = 1617 \left(1 - \frac{1}{3} \right) \left(1 - \frac{1}{7} \right) \left(1 - \frac{1}{11} \right) = 840.$$ 

Hence the group $U(1617) \simeq U(3) \oplus U(7^2) \oplus U(11)$. Using Gauss’s result, we have

$$U(1617) \simeq U(3) \oplus U(7^2) \oplus U(11) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_{42} \oplus \mathbb{Z}_{10}.$$ 

4. Let $G$ be a group and let $H$ be a subgroup of $G$. Let $a \in G$. Prove that $aH = H$ if and only if $a \in H$.

Answer: Suppose $aH = H$. Then $a = ae \in aH = H$.

Next, suppose $a \in H$. We want to show $aH = H$ (that is, $aH \subseteq H$ and $H \subseteq aH$).

Since $a \in H$ and $H$ is closed under group operation, therefore $aH \subseteq H$. Now we prove that $H \subseteq aH$. Let $h \in H$. Then $h = eh = (aa^{-1})h = a(a^{-1}h) \in aH$. Hence $H \subseteq aH$. Therefore if $a \in H$, then $aH = H$.

5. Let $G$ be a group of order 60. What are the possible orders for the subgroups of $G$? Is there a subgroup of order 11 in $G$?

Answer: Since the Lagrange’s Theorem says that the order of a subgroup divide the order of the group $G$, therefore any divisor $k$ of the order of the group $G$ may yield a possible subgroup of order $k$. Since the divisors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60, therefore $G$ has subgroups of possible orders 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60.

There is no subgroup of order 11 since 11 does not divide 60.

6. Find all the left cosets of the subgroup $H = \{(0, 1), (1, 2), (2, 4), (3, 3)\}$ in $\mathbb{Z}_4 \oplus U(5)$.

Answer: The order of the group $\mathbb{Z}_4 \oplus U(5)$ is $4 \phi(5)$ which is 16. The order of the group $H$ is 4. Hence by Lagrange’s Theorem there are 4 distinct cosets of $H$ in $\mathbb{Z}_4 \oplus U(5)$. These cosets are

$$(0, 1) \ast H = \{(0, 1), (1, 2), (2, 4), (3, 3)\}, \quad (1, 1) \ast H = \{(1, 1), (2, 2), (3, 4), (0, 3)\},$$

$$(2, 1) \ast H = \{(2, 1), (3, 2), (0, 4), (1, 3)\}, \quad (3, 1) \ast H = \{(3, 1), (0, 2), (1, 4), (2, 3)\}.$$ 

The other 12 cosets can be computed similarly.
7. Prove that the center of a group \( G \) is a normal subgroup of \( G \).

**Answer:** Let \( G \) be a group, and \( Z(G) \leq G \). We want to show that \( Z(G) \triangleleft G \). Let \( ghg^{-1} \in gZ(G)g^{-1} \). We want to show that \( ghg^{-1} \in Z(G) \). Since

\[
ghg^{-1} = gg^{-1}h \quad \text{(because } h \in Z(G)\text{)}
\]

\[
= h 
\]

\[
\in Z(G),
\]

we have \( gZ(G)g^{-1} \subseteq Z(G) \). Hence by normal subgroup test, we get \( Z(G) \triangleleft G \).

8. Prove that \( SL(2, \mathbb{R}) \) is a normal subgroup of the group \( GL(2, \mathbb{R}) \).

**Answer:** Let us denote \( G = GL(2, \mathbb{R}) \) and \( H = SL(2, \mathbb{R}) \). We want to show \( H \triangleleft G \). That is, \( xHx^{-1} \subseteq H \) for all \( x \in G \). Let \( x \in G \). Then \( x \) will be a 2-by-2 matrix of the form

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

with \( \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \). Let \( h \) be an element in \( H \). Then \( h \) is a matrix of the form

\[
\begin{pmatrix}
e & f \\
g & h
\end{pmatrix}
\]

with \( \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \). Consider the element \( xhx^{-1} \in xHx^{-1} \). Hence

\[
\det (xHx^{-1}) = \det \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right)
\]

\[
= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} e & f \\ g & h \end{pmatrix} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}
\]

\[
= \det \begin{pmatrix} e & f \\ g & h \end{pmatrix} = 1
\]

and thus \( xHx^{-1} \in H \). Therefore \( H \triangleleft G \).

9. What is the subgroup generated by the element \((2, 2)\) in the group \( \mathbb{Z} \oplus \mathbb{Z} \)? What is the order of the factor group \( (\mathbb{Z} \oplus \mathbb{Z})/ \langle (2, 2) \rangle \)? Is this factor group cyclic?

**Answer:** Let \( H = \langle (2, 2) \rangle \). Then

\[
H = \langle (2, 2) \rangle
\]

\[
= \{ n(2, 2) \mid n \in \mathbb{Z} \}
\]

\[
= \{ (2n, 2n) \mid n \in \mathbb{Z} \}.
\]

Consider the subset \( K \) of distinct cosets of \( H \) in \( \mathbb{Z} \oplus \mathbb{Z} \) given by

\[
K = \{ (x, 0) + \langle (2, 2) \rangle \mid x \in \mathbb{Z} \} = \{ (x, 0) + H \mid x \in \mathbb{Z} \}.
\]

Then \( K \subseteq (\mathbb{Z} \oplus \mathbb{Z})/ \langle (2, 2) \rangle \) and \( |K| = \infty \). Therefore \( |(\mathbb{Z} \oplus \mathbb{Z})/ \langle (2, 2) \rangle | = \infty \).
Consider the coset $(1, 1) + \langle (2, 2) \rangle \in (\mathbb{Z} \oplus \mathbb{Z})/\langle (2, 2) \rangle$. Since
\[
2 ((1, 1) + \langle (2, 2) \rangle) = 2(1, 1) + H = (2, 2) + H = H = (0, 0) + H,
\]
the order of the element $(1, 1) + \langle (2, 2) \rangle$ is 2. The infinity order group $(\mathbb{Z} \oplus \mathbb{Z})/\langle (2, 2) \rangle$ can not be cyclic since it has an element $(1, 1) + \langle (2, 2) \rangle$ of order 2.

10. What is the subgroup generated by the element $(4, 2)$ in the group $\mathbb{Z} \oplus \mathbb{Z}$? What is the order of the factor group $(\mathbb{Z} \oplus \mathbb{Z})/\langle (4, 2) \rangle$? Is this factor group cyclic?

**Answer:** Let $H = \langle (4, 2) \rangle$. Then
\[
H = \langle (4, 2) \rangle = \{ n(4, 2) \mid n \in \mathbb{Z} \} = \{ (4n, 2n) \mid n \in \mathbb{Z} \}.
\]

Consider the subset $K$ of distinct cosets of $H$ in $\mathbb{Z} \oplus \mathbb{Z}$ given by
\[
K = \{ (x, 0) + \langle (4, 2) \rangle \mid x \in \mathbb{Z} \} = \{ (x, 0) + H \mid x \in \mathbb{Z} \}.
\]

Then $K \subset (\mathbb{Z} \oplus \mathbb{Z})/\langle (4, 2) \rangle$ and $|K| = \infty$. Therefore $|(\mathbb{Z} \oplus \mathbb{Z})/\langle (4, 2) \rangle| = \infty$.

Consider the coset $(6, 3) + \langle (4, 2) \rangle \in (\mathbb{Z} \oplus \mathbb{Z})/\langle (4, 2) \rangle$. Since
\[
2 ((6, 3) + \langle (4, 2) \rangle) = 2(6, 3) + H = (12, 6) + H = H = (0, 0) + H,
\]
the order of the element $(6, 3) + \langle (4, 2) \rangle$ is 2. The infinity order group $(\mathbb{Z} \oplus \mathbb{Z})/\langle (4, 2) \rangle$ can not be cyclic since it has an element $(6, 3) + \langle (4, 2) \rangle$ of order 2.