Direction: You are required to complete this test within 50 minutes. In order to receive full credit, answer each problem completely and must show all work. Good Luck!

1. (13 points) Prove by the method of induction that for every positive integer \( n \),
   
   \[
   1 + 4 + 7 + 10 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}.
   \]

   **Answer:** Let \( S = \{ k \in \mathbb{N} \mid 1 + 4 + 7 + 10 + \cdots + (3k - 2) = \frac{k(3k-1)}{2} \} \). Clearly \( 1 \in S \) since \( 1 = \frac{1(3-1)}{2} \). Suppose \( n \in S \), that is \( 1 + 2 + \cdots + (3n - 2) = \frac{n(n+1)}{2} \). We want to show that \( n + 1 \in S \). For this we calculate
   
   \[
   1 + 2 + \cdots + (3n - 2) + (3n + 1) = \frac{n(3n-1)}{2} + (3n + 1) \quad \text{(by induction hypothesis)}
   \]
   
   \[
   = \frac{3n^2 + 5n + 2}{2}
   \]
   
   \[
   = \frac{(n+1)(3n+2)}{2}
   \]
   
   \[
   = \frac{(n+1)(3(n+1)-1)}{2}.
   \]

   Hence \( n + 1 \in S \), and by induction principle, we have \( S = \mathbb{N} \).

2. (12 points) Let \( U(5) \) denote the set of all positive integers less than 5 and relatively prime to 5. Verify that \( U(5) \) is a group under multiplication modulo 5. What is the Cayley table for this group?

   **Answer:** Since \( U(5) \) denotes the set of all positive integers less than 5 and relatively prime to 5, thus
   
   \[
   U(5) = \{ 1, 2, 3, 4 \}.
   \]

   Next we verify that \( (U(5), \cdot) \) is a group. The identity of \( (U(5), \cdot) \) is 1. Since multiplication of integers is associative, therefore the associativity law holds in \( (U(5), \cdot) \). Each element in \( (U(5), \cdot) \) has a multiplicative inverse. We list the inverse of each element in the table below:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^{-1} )</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>
The group $(U(5), \cdot)$ is closed under multiplication which can be seen from the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
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<td>2</td>
<td>2</td>
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<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

3. (13 points) What is the inverse of the group element \( \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} \) in $GL(2, \mathbb{Z}_{11})$?

**Answer:** The determinant of the matrix \( \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} \) is

\[
\det \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} = 10 - 18 = -8 = 3 \mod 11.
\]

The multiplicative inverse of 3 is 4 since $3 \cdot 4 = 1 \mod 11$. The inverse of \( \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} \) is given by

\[
\begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -6 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 5 \cdot 4 & -6 \cdot 4 \\ -3 \cdot 4 & 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ -1 & 8 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 10 & 8 \end{pmatrix}.
\]

4. (12 points) What is the inverse of the group element \( \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \) in $SL(2, \mathbb{Z}_7)$?

**Answer:** The determinant of the matrix \( \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \) is

\[
\det \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = 1.
\]

The multiplicative inverse of 1 is 1 since $1 \cdot 1 = 1 \mod 7$. The inverse of \( \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \) is given by

\[
\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix}.
\]

5. (13 points) Find the symmetry group of the logo and write out a multiplication table for this group.

**Answer:** The symmetric group of transformations for this logo consists of rotations by $0^\circ$, $90^\circ$, $180^\circ$ and $270^\circ$. We denote these by $R_0$, $R_{90}$, $R_{180}$ and $R_{270}$. Hence the symmetric group of this logo is $(G, \cdot)$, where

\[
G = \{ R_0, R_{90}, R_{180}, R_{270} \}.
\]
The multiplication table is

<table>
<thead>
<tr>
<th></th>
<th>$R_0$</th>
<th>$R_{90}$</th>
<th>$R_{180}$</th>
<th>$R_{270}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_0$</td>
<td>$R_0$</td>
<td>$R_{90}$</td>
<td>$R_{180}$</td>
<td>$R_{270}$</td>
</tr>
<tr>
<td>$R_{90}$</td>
<td>$R_{90}$</td>
<td>$R_{180}$</td>
<td>$R_{270}$</td>
<td>$R_0$</td>
</tr>
<tr>
<td>$R_{180}$</td>
<td>$R_{180}$</td>
<td>$R_{270}$</td>
<td>$R_0$</td>
<td>$R_{90}$</td>
</tr>
<tr>
<td>$R_{270}$</td>
<td>$R_{270}$</td>
<td>$R_0$</td>
<td>$R_{90}$</td>
<td>$R_{180}$</td>
</tr>
</tbody>
</table>

6. **(12 points)** Show that in an abelian group, $(ab)^5 = a^5b^5$.

**Answer:** Suppose the group is abelian, that is $ab = ba$ for all $a, b \in G$.

$$
(ab)^5 = (ab)(ab)(ab)(ab)(ab)
= a(ba)(ba)(ba)b
= a(ab)(ab)(ab)b
= a(a(ba)(ba)b)b
= a(aa(ba)b)b
= a(aa(a(ba)b)b)b
= a(aa(a(ba)b)b)b
= a^5b^5.
$$

7. **(13 points)** Let $G = \{ x \in \mathbb{R} \mid x \neq 1 \}$ and $a \star b = a + b - ab$ for $a, b \in G$. Then $(G, \star)$ is a group. (a) Find the identity element of this group. (b) If $a$ is an element in $G$, then find its inverse.

**Answer:** (a) The identity of $(G, \star)$ is 0 since

$$a \star 0 = 0 \star a = a.$$

(b) The inverse of any element $a \in G$ is an element $b$ such that

$$a \star b = 0.$$

Hence

$$a + b - ab = 0$$
and

\[ a + b(1 - a) = 0. \]

Therefore

\[ b = \frac{a}{a - 1}. \]

8. **(12 points)** In a dihedral group \( D_4 \), explain why a reflection followed by another reflection is a rotation.

**Answer:** In a dihedral group, a reflection fixes points on the axis of reflection and changes the position of every other point. Hence two successive distinct reflections leaves the center fixed. Therefore it is a rotation.

9. **(13 points)** Let \( \mathbb{Z} \) be the set of integers. For \( a, b \in \mathbb{Z} \), define \( a \sim b \) as \( a = b \mod 3 \). Show that \( \sim \) is an equivalence relation on \( \mathbb{Z} \). What is the quotient space \( \mathbb{Z}/\sim \)?

**Answer:** Clearly \( \sim \) is an equivalence relation (we did some thing similar to this in class). Now we compute the following equivalence classes:

\[
[0] = \{ x \in \mathbb{Z} \mid x \sim 0 \} = \{ x \in \mathbb{Z} \mid 3/(x - 0) \} = \{ x \in \mathbb{Z} \mid x = 3k, k \in \mathbb{Z} \} = \{ 3k \mid k \in \mathbb{Z} \} = \{ \cdots, -9, -6, -3, 0, 3, 6, 9, \cdots \} = 3\mathbb{Z} + 0,
\]

\[
[1] = \{ x \in \mathbb{Z} \mid x \sim 1 \} = \{ x \in \mathbb{Z} \mid 3/(x - 1) \} = \{ x \in \mathbb{Z} \mid x = 3k + 1, k \in \mathbb{Z} \} = \{ 3k + 1 \mid k \in \mathbb{Z} \} = \{ \cdots, -8, -5, -2, 1, 4, 7, 10, \cdots \} = 3\mathbb{Z} + 1,
\]

and

\[
[2] = \{ 3k + 2 \mid k \in \mathbb{Z} \} = \{ \cdots, -7, -4, -1, 2, 5, 8, 11, \cdots \} = 3\mathbb{Z} + 2
\]

Hence the quotient space \( \mathbb{Z}/\sim \) is given by

\[ \mathbb{Z}/\sim = \{ [0], [1], [2] \}. \]

10. **(12 points)** Let \( U(11) \) denote the set of all positive integers less than 11 and relatively prime to 11. We now know that it is a group under multiplication modulo 11. Find all the elements in this group \( U(11) \) that satisfy \( x^2 = 1 \).

**Answer:**

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>1</td>
<td>2</td>
<td>9</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>9</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Hence the solution of \( x^2 = 1 \) in \( U(11) \) is \{ 1, 10 \}. 