1. Consider the function $\phi : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_6$ defined by $\phi(x) = x \mod 6$. (a) Is $\phi$ a ring homomorphism? (b) What is the kernel of $\phi$? (c) What are the elements of the set $\phi(\mathbb{Z}_{12})$? (d) What are the elements of the factor ring $\mathbb{Z}_{12}/\text{Ker}\phi$? (e) Is $\mathbb{Z}_{12}/\text{Ker}\phi$ isomorphic to $\phi(\mathbb{Z}_{12})$?

Answer: (a) Since $\phi(x + y) = (x + y) \mod 6 = x \mod 6 + y \mod 6 = \phi(x) + \phi(y)$, and $\phi(xy) = (xy) \mod 6 = (x \mod 6)(y \mod 6) = \phi(x)\phi(y)$, therefore $\phi : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_6$ is a ring homomorphism.

(b) The Kernel of $\phi$ is given by
$$\text{Ker}\phi = \{x \in \mathbb{Z}_{12} \mid \phi(x) = 0\} = \{x \in \mathbb{Z}_{12} \mid x \mod 6 = 0\} = \{0, 6\} = \langle 6 \rangle.$$

(c) The image of $\mathbb{Z}_{12}$ under the mapping $\phi$ is given by
$$\phi(\mathbb{Z}_{12}) = \{\phi(x) \mid x \in \mathbb{Z}_{12}\} = \{x \mod 6 \mid x \in \mathbb{Z}_{12}\} = \{0, 1, 2, 3, 4, 5\} = \mathbb{Z}_6.$$

(d) The factor ring $\mathbb{Z}_{12}/\text{Ker}\phi$ is given by
$$\mathbb{Z}_{12}/\text{Ker}\phi = \mathbb{Z}_{12}/\langle 6 \rangle = \{0 + \langle 6 \rangle, 1 + \langle 6 \rangle, 2 + \langle 6 \rangle, \ldots, 5 + \langle 6 \rangle\}.$$

(e) Define the mapping $\psi : \mathbb{Z}_{12}/\langle 6 \rangle \rightarrow \mathbb{Z}_6$ by $\psi(x + \langle 6 \rangle) = x$ for each $x \in \mathbb{Z}_6$. Then $\psi$ is one-to-one, onto and preserves the ring operations. Therefore $\psi$ is an isomorphism and $\mathbb{Z}_{12}/\text{Ker}\phi \simeq \phi(\mathbb{Z}_{12})$.

2. Let $\phi : \mathbb{Z} \rightarrow \mathbb{Z}_3 \oplus \mathbb{Z}_4$ be a function defined by $\phi(x) = (x \mod 3, x \mod 4)$. (a) Is $\phi$ a ring homomorphism? (b) What is the kernel of $\phi$? (c) What are the elements of the set $\phi(\mathbb{Z})$? (d) What are the elements of the factor ring $\mathbb{Z}/\text{Ker}\phi$? (e) Is $\mathbb{Z}/\text{Ker}\phi$ isomorphic to $\phi(\mathbb{Z})$?

Answer: (a) Since
$$\phi(x + y) = (x + y \mod 3, x + y \mod 4)$$
$$= (x \mod 3, x \mod 4) + (y \mod 3, y \mod 4)$$
$$= \phi(x) + \phi(y)$$

and
$$\phi(xy) = (xy \mod 3, xy \mod 4)$$
$$= (x \mod 3, x \mod 4)(y \mod 3, y \mod 4)$$
$$= \phi(x)\phi(y),$$

therefore $\phi : \mathbb{Z} \rightarrow \mathbb{Z}_3 \oplus \mathbb{Z}_4$ is a ring homomorphism.
(b) The Kernel of $\phi$ is given by

$$Ker \phi = \{ x \in \mathbb{Z} \mid \phi(x) = (0, 0) \} = \{ x \in \mathbb{Z} \mid x \mod 3 = 0 \mod 4 \} = \langle 12 \rangle = 12\mathbb{Z}.$$ 

(c) The image of $\mathbb{Z}$ under the mapping $\phi$ is given by

$$\phi(\mathbb{Z}) = \{ \phi(x) \mid x \in \mathbb{Z} \} = \{ (x \mod 3, \ x \mod 4) \mid x \in \mathbb{Z} \} = \mathbb{Z}_3 \oplus \mathbb{Z}_4.$$ 

(d) The factor ring $\mathbb{Z}/Ker \phi$ is given by

$$\mathbb{Z}/Ker \phi = \mathbb{Z}/12\mathbb{Z} = \{ 0 + 12\mathbb{Z}, 1 + 12\mathbb{Z}, 2 + 12\mathbb{Z}, \ldots, 11 + 12\mathbb{Z} \}.$$ 

(e) Define the mapping $\psi : \mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}_3 \oplus \mathbb{Z}_4$ by $\psi(x + 12\mathbb{Z}) = x$ for each $x \in \mathbb{Z}_{12}$. Then $\psi$ is one-to-one, onto and preserves the ring operations. Therefore $\psi$ is an isomorphism and $\mathbb{Z}_{12}/Ker \phi \cong \phi(\mathbb{Z})$.

3. Determine all ring homomorphisms from the ring $(\mathbb{Z}, +, \cdot)$ to $(\mathbb{Z}, +, \cdot)$. Also, determine all ring homomorphisms from the ring $(\mathbb{R}, +, \cdot)$ to $(\mathbb{R}, +, \cdot)$. 

Answer: Since $f : \mathbb{Z} \to \mathbb{Z}$ is a ring homomorphism, therefore we have $f(n) = nf(1)$ and $f(1) = f(1)^2$. Renaming $f(1)$ as $a$, we get $f(n) = an$ for all $n \in \mathbb{Z}$. Since $a = a^2$, therefore either $a = 0$ or $a = 1$. Hence the ring homomorphisms from the ring $\mathbb{Z}$ to $\mathbb{Z}$ are $f(n) = n$ and $f(n) = 0$ for all $n \in \mathbb{Z}$.

Finding all the homomorphisms from the rings $\mathbb{R}$ to $\mathbb{R}$ is a bit hard. Here are the essential steps one needs to establish for finding these ring homomorphisms. First, show that $f(x + y) = f(x) + f(y)$ implies $f(x) = xf(1)$ for all $x \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Second, use the fact $f(xy) = f(x)f(y)$ to show $f(t) \geq 0$ for all $t \geq 0$. Using this fact and $f(x + y) = f(x) + f(y)$, show $f(x)$ is an increasing function. Third, show every increasing function that satisfies $f(x + y) = f(x) + f(y)$ is linear, that is $f(x) = xf(1)$ for all $x \in \mathbb{R}$. Fourth, use $f(xy) = f(x)f(y)$ to show $f(1) = 0$ or $f(1) = 1$. Hence there are two ring homomorphisms from the ring $\mathbb{R}$ to $\mathbb{R}$, and they are $f(x) = x$ and $f(x) = 0$. The detail proof can be found in a manuscript due to Sahoo (1997).

4. Let $\phi : \mathbb{R}[x] \to \mathbb{C}$ be a function defined by $\phi(f(x)) = f(i)$. Here $\mathbb{R}$ denotes the ring of real numbers, $\mathbb{C}$ denotes the ring of complex numbers and $i$ denotes $\sqrt{-1}$. (a) Is $\phi$ a ring homomorphism? (b) What is the kernel of $\phi$? (c) What is the set $\phi(\mathbb{R}[x])$? (d) What is the factor ring $\mathbb{R}[x]/Ker \phi$? (e) Is $\mathbb{R}[x]/Ker \phi$ isomorphic to $\phi(\mathbb{R}[x])$? (Hint: see example 3 on page 298.)

Answer: (a) Since 

$$\phi((f + g)(x)) = (f + g)(i) = f(i) + g(i) = \phi(f(x)) + \phi(f(y))$$

and 

$$\phi((fg)(x)) = (fg)(i) = f(i)g(i) = \phi(f(x))\phi(f(y)),$$

therefore $\phi : \mathbb{R}[x] \to \mathbb{C}$ is a ring homomorphism.
(b) The Kernel of \( \phi \) is given by

\[
\ker \phi = \{ f(x) \in \mathbb{R}[x] \mid f(i) = 0 \} = \{ (x^2 + 1)g(x) \mid g(x) \in \mathbb{R}[x] \} = \langle x^2 + 1 \rangle.
\]

(Note that \( f(i) = 0 \) implies \( f(x) = (x^2 + 1)g(x) \) for some \( g(x) \in \mathbb{R}[x] \).)

(c) The image of \( \mathbb{R}[x] \) under the mapping \( \phi \) is given by

\[
\phi(\mathbb{R}[x]) = \{ \phi(f(x)) \mid f(x) \in \mathbb{R}[x] \} = \{ f(i) \mid f(x) \in \mathbb{R}[x] \}
\]

\[
= \left\{ \sum_{k=0}^{n} a_k (i)^k \mid a_k \in \mathbb{R} \right\}
\]

\[
= \{ A + Bi \mid A, B \in \mathbb{R} \}
\]

\[= \mathbb{C}.
\]

(d) The factor ring \( \mathbb{R}[x]/\ker \phi \) is given by

\[
\mathbb{R}[x]/\ker \phi = \mathbb{R}[x]/\langle x^2 + 1 \rangle = \{ ax + b + < x^2 + 1 > \mid a, b \in \mathbb{R} \}.
\]

(e) Define the mapping \( \psi : \mathbb{R}[x]/\langle x^2 + 1 \rangle \to \mathbb{C} \) by \( \psi(ax + b + < x^2 + 1 >) = ai + b \) for \( a, b \in \mathbb{R} \). Then \( \psi \) is one-to-one, onto and preserves the ring operations. Therefore \( \psi \) is an isomorphism and \( \mathbb{R}[x]/\ker \phi \cong \phi(\mathbb{R}[x]) \).

5. Let \( f(x) = 4x^3 + 2x^2 + x + 3 \) and \( g(x) = 3x^4 + 3x^3 + 3x^2 + x + 4 \) be two polynomials in the ring \( \mathbb{Z}_5[x] \). Compute \( f(x) + g(x) \) and \( f(x)g(x) \).

**Answer:** \( f(x) + g(x) = 3x^4 + 2x^3 + 2x + 2 \) and \( f(x)g(x) = 2x^7 + 3x^6 + x^5 + 2x^4 + 3x^2 + 2x + 2 \).

6. Find all the zeros of the polynomial \( f(x) = x^2 + 3x + 2 \) in \( \mathbb{Z}_6 \). Find the zeros of the same polynomial in \( \mathbb{Z}_{11} \).

**Answer:** Since \( f(x) = x^2 + 3x + 2 \), and

<table>
<thead>
<tr>
<th>( x \in \mathbb{Z}_6 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) \mod 6 )</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

the zeros of \( f(x) \) in \( \mathbb{Z}_6 \) are 1, 2, 4, 5. Similarly, the zeros of \( f(x) \) in \( \mathbb{Z}_{11} \) are 9 and 10.

7. Let \( f(x) = 5x^4 + 3x^3 + 1 \) and \( g(x) = 3x^2 + 2x + 1 \) be two polynomials in the ring \( \mathbb{Z}_7[x] \). Determine the quotient and the remainder upon dividing \( f(x) \) by \( g(x) \).

**Answer:** If \( f(x) = 5x^4 + 3x^3 + 1 \) and \( g(x) = 3x^2 + 2x + 1 \), then

\[
f(x) = q(x)g(x) + r(x)
\]

in \( \mathbb{Z}_7[x] \) whenever \( q(x) = 4x^2 + 3x + 6 \) and \( r(x) = 6x + 2 \).

8. Let \( f(x) \) be a polynomial in \( \mathbb{R}[x] \). Prove that \( a \) is a root of a polynomial \( f(x) \) if and only if \( x - a \) is a factor of \( f(x) \). (Hint: use Remainder Theorem.)
Answer: By Remainder Theorem, we have

\[ f(x) = (x-a)g(x) + f(a), \]  

where \( g(x) \in F[x] \). Suppose \( a \) is a zero of \( f(x) \). Then \( f(a) = 0 \). Hence from (1), we get \( f(x) = (x-a)g(x) \). This proves that \( (x-a) \) is factor of \( f(x) \).

Conversely, if \( (x-a) \) is factor of \( f(x) \), then the remainder \( f(a) \) must be zero. That is \( f(a) = 0 \) and thus \( a \) is a zero of \( f(x) \).

9. A polynomial \( f(x) \) is said to be irreducible in the ring \((\mathbb{Z}_5[x], +, \cdot)\) if it has no zeros in \( \mathbb{Z}_5 \). Which of the following polynomials
\[ f(x) = 2x^3 + x^2 + 4x + 1; \quad g(x) = x^4 + 2; \quad h(x) = x^4 + 4x^3 + 2 \]
are irreducible in \( \mathbb{Z}_5[x] \).

Answer: Since \( f(x) = 2x^3 + x^2 + 4x + 1 \), and

<table>
<thead>
<tr>
<th>( x \in \mathbb{Z}_5 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) \ mod 5 )</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

therefore \( f(x) \) is irreducible in \( \mathbb{Z}_5[x] \).

Since \( g(x) = x^4 + 2 \), and

<table>
<thead>
<tr>
<th>( x \in \mathbb{Z}_5 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) \ mod 5 )</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

therefore \( g(x) \) is irreducible in \( \mathbb{Z}_5[x] \).

Since \( h(x) = x^4 + 4x^3 + 2 \), and

<table>
<thead>
<tr>
<th>( x \in \mathbb{Z}_5 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(x) \ mod 5 )</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

therefore \( h(x) \) is not irreducible in \( \mathbb{Z}_5[x] \).

10. Determine how many irreducible quadratics are there in \( \mathbb{Z}_5[x] \). Quadratics are polynomials of degree 2.

Answer: The number of quadratic polynomials of the form \( ax^2 + bx + c \) where \( a, b, c \in \mathbb{Z}_5 \) and \( a \neq 0 \) is equal to \( (4)(5)(5) = 100 \).

The number of reducible polynomials in \( \mathbb{Z}_5 \) is equal to the number of distinct expressions of the form \( a(x + \alpha)(x + \beta) \) and \( a(x + \alpha)^2 \) for \( \alpha, \beta \in \mathbb{Z}_5 \).

In \( \mathbb{Z}_5 \), the number of distinct expressions of the form \( (x + \alpha)(x + \beta) \) is equal to \( \binom{5}{2} = 10 \). Hence the number of distinct expressions of the form \( a(x + \alpha)(x + \beta) \) is equal to \( 4(10) = 40 \).

Similarly, the number of expressions of the form \( a(x + \alpha)^2 \) is \( 4(5) = 20 \).

Thus, the total number of reducible polynomials in \( \mathbb{Z}_5 \) is equal to \( 40 + 20 = 60 \). Hence the number of irreducible quadratics of the form \( ax^2 + bx + c \) is equal to \( 100 - 60 = 40 \).